Math 211 - Bonus Exercise 4 (please discuss on Forum)

1) Construct short exact sequences

$$1 \to \mathbb{Z}/2\mathbb{Z} \to D_8 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 1$$

$$1 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to D_8 \to \mathbb{Z}/2\mathbb{Z} \to 1$$

explicitly (i.e. explicitly describe all the homomorphisms involved).

2) Given short exact sequences $1 \to K \xrightarrow{f} G \xrightarrow{g} L \to 1$ and $1 \to K' \xrightarrow{f'} G' \xrightarrow{g'} L' \to 1$, a homomorphism between them is a diagram of vertical homolorphisms

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\alpha"'] & G \arrow[r, "g"] \arrow[d, "\beta"'] & L \arrow[r] \arrow[d, "\gamma"'] & 1 \\
1 \arrow[r] & K' \arrow[r, "f'"] & G' \arrow[r, "g'"] & L' \arrow[r] & 1
\end{tikzcd}$$

for which both squares commute. Show that

- If α and γ are injective, then so is β.
- If α and γ are surjective, then so is β.
- If β is injective, then α is injective.
- If β is injective and α is surjective, then γ is injective.
- If β is surjective, then γ is surjective.
- If β is surjective and γ is injective, then α is surjective.

3) Suppose that a group G contains subgroups K and L such that K is normal and $K \cap L = \{e\}$. Then show that G contains a subgroup isomorphic to $K \rtimes L$.

4) Consider the semidirect product $G = K \rtimes L$ corresponding to a certain action by automorphisms $L \curvearrowright K$. You may regard L as a subgroup of G (see the previous exercise) so you have an induced adjoint action

$$L \curvearrowright G$$

- Show that the action above preserves the subgroup $K \trianglelefteq G$, and that the resulting action of L on K matches the action $L \curvearrowright K$ that we used to build up our semidirect product.
- Compute the centralizer and the normalizer of the subgroup $K \trianglelefteq G$ with respect to the action above.

5) Consider any abelian group K. Show that the assignment $k \mapsto -k$ gives rise to an action $\mathbb{Z}/2\mathbb{Z} \curvearrowright K$ by automorphisms, hence a semidirect product $K \rtimes \mathbb{Z}/2\mathbb{Z}$. What is this semidirect product in the simplest possible case $K = \mathbb{Z}/n\mathbb{Z}$?